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TRIAL LOAD ANALYSIS OF STRESSES IN DAMS

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TRIAL LOAD ANALYSIS OF STRESSES IN DAMS

Otto Pfafstetter[1]

SYNOPSIS

In this paper the author presents a general analytical proof for the trial load analysis of stresses in concrete dams. The presentation also endeavors to confirm the validity of the use of the principal systems in the trial load analysis, namely: vertical cantilevers, horizontal arches or beams, and the twisted structure. The general equations developed for arch dams are simplified for the particular analysis of straight gravity dams having grouted and ungrouted joints.

Presented separately are the two methods of performing the analysis: the use of self-balancing loads, and the use of twisted structures.

The Use of Self-balancing Loads in the Analysis of Arch Dams

Figure 1 illustrates an arch dam element limited by two horizontal sections, two radial vertical sections, and upstream and downstream faces. Chosen for the purposes of analysis in this paper is a rectangular coordinate system having X and Y axes parallel, respectively, to tangential and radial directions in the center of the arch element, and the Z axis in the vertical direction. The center of the arch element may be defined by the intersection of the line connecting the gravity centers of the two horizontal sections with the horizontal plane passing through the gravity centers of the radial sections.

In Figure 1 are the following geometrical elements:

a = Horizontal distance between gravity centers of the horizontal sections;

b = Horizontal distance between gravity centers of the radial sections;

c = Horizontal distance from the arch element center to the line connecting the gravity centers of the radial sections;

d = Vertical distance between gravity centers of the horizontal sections;

ϕ = Angle between radial sections.

The resultant of all external forces as water load, concrete weight, and earthquake forces can be referred to the center of the arch element, leading to the following components:

X_o, Y_o, Z_o - force components along the three axes,

M_{xo}, M_{yo}, M_{zo} - moment components related to lines parallel to the coordinate axis, drawn through the arch element center.

By integrating all stresses along a horizontal section a resultant force is obtained in some position in space, which can be replaced by the following forces and moments acting on the gravity center of this section:

N_z - normal force,

V_{zx} - shear in X direction,

M_{zx}, M_{zy}, M_{zz} - moments related to lines parallel to the coordinate axis, drawn through the gravity center of the horizontal section.

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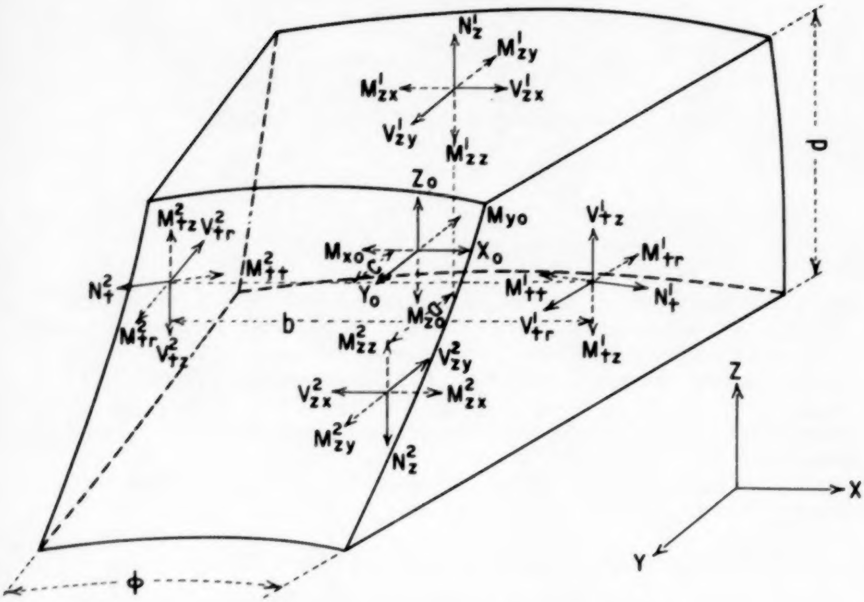

FIG. I - ARCH DAM ELEMENT

Referring, in the same way, the resultant of all stresses on a radial section to the gravity center of this section, we have the following force and moment components:

N_t - tangential (normal) force,
V_{tr} - radial shear,
V_{tz} - shear in vertical direction,
M_{tt}, M_{tr}, M_{tz} - moments related respectively to tangential, radial, and vertical lines, drawn through the gravity center of the radial section.

The moments are represented by clockwise vectors (in dashed lines) normal to their respective planes.

The directions of the arrows indicated in Figure 1 give the positive value for the forces and moments.

The six equilibrium conditions of the arch element to translation along the three axes and to rotation around lines parallel to the coordinate axis, drawn through the center of the element, lead to:

$$(1)\quad X_o + V^1_{zx} - V^2_{zx} + \cos\frac{\Phi}{2}\left(N^1_t - N^2_t\right) - \sin\frac{\Phi}{2}\left(V^1_{tr} + V^2_{tr}\right) = 0$$

$$(2)\quad Y_o + V^1_{zy} - V^2_{zy} + \cos\frac{\Phi}{2}\left(V^1_{tr} - V^2_{tr}\right) + \sin\frac{\Phi}{2}\left(N^1_t + N^2_t\right) = 0$$

(3) $Z_o + N^1_z - N^2_z + V^1_{tz} - V^2_{tz} = 0$

$$M_{xo} + M^1_{zx} - M^2_{zx} + \frac{d}{2}(V^1_{zy} + V^2_{zy}) + \frac{a}{2}(N^1_z + N^2_z) + \cos\frac{\Phi}{2}$$

(4)

$$(M^1_{tt} - M^2_{tt}) - \sin\frac{\Phi}{2}(M^1_{tr} + M^2_{tr}) - c\,(V^1_{tz} - v^2_{tz}) = 0$$

(5) $M_{yo} + M^1_{zy} - M^2_{zy} - \frac{d}{2}(V^1_{zx} + V^2_{zx}) + \cos\frac{\Phi}{2}(M^1_{tr} - M^2_{tr})$

$$+ \sin\frac{\Phi}{2}(M^1_{tt} + M^2_{tt}) + \frac{b}{2}(V^1_{tz} + V^2_{tz}) = 0$$

(6) $M_{zo} + M^1_{zz} - M^2_{zz} - \frac{a}{2}(V^1_{zx} + V^2_{zx}) + M^1_{tz} - M^2_{tz}$

$$+ c \cdot \cos\frac{\Phi}{2}(N^1_t - N^2_t) - \frac{b}{2} \cdot \sin\frac{\Phi}{2}(N^1_t - N^2_t)$$

$$- \frac{b}{2} \cdot \cos\frac{\Phi}{2}(V^1_{tr} + V^2_{tr}) - c \cdot \sin$$

$$\frac{\Phi}{2}(V^1_{tr} + V^2_{tr}) = 0$$

As the center of the arch element is not equidistant in vertical and horizontal direction from the gravity centers of the horizontal sections, the distances $\frac{a}{2}$ and $\frac{d}{2}$ used for the forces in equations (4), (5), and (6) are not exact. However, the difference can be neglected in analyzing finite cantilever elements and especially if they are infinitesimal in the vertical direction.

To simplify equations (1) to (6), set:

$N^1_z - N^2_z = \Delta N_z$	$M^1_{zx} - M^2_{zx} = \Delta M_{zx}$
$V^1_{zx} - V^2_{zx} = \Delta V_{zx}$	$M^1_{zy} - M^2_{zy} = \Delta M_{zy}$
$V^1_{zy} - V^2_{zy} = \Delta V_{zy}$	$M^1_{zz} - M^2_{zz} = \Delta M_{zz}$
$N^1_t - N^2_t = \Delta N_t$	$M^1_{tt} - M^2_{tt} = \Delta M_{tt}$
$V^1_{tr} - V^2_{tr} = \Delta V_{tr}$	$M^1_{tr} - M^2_{tr} = \Delta M_{tr}$
$V^1_{tz} - V^2_{tz} = \Delta V_{tz}$	$M^1_{tz} - M^2_{tz} = \Delta M_{tz}$

$$1/2\,(N^1{}_z + N^2{}_z) = N_z \qquad 1/2\,(M^1{}_{tt} + M^2{}_{tt}) = M_{tt}$$

$$1/2\,(V^1{}_{zx} + V^2{}_{zx}) = V_{zx} \qquad 1/2\,(M^1{}_{tr} + M^2{}_{tr}) = M_{tr}$$

$$1/2\,(V^1{}_{zy} + V^2{}_{zy}) = V_{zy}$$

$$1/2\,(N^1{}_t + N^2{}_t) = N_t$$

$$1/2\,(V^1{}_{tr} + V^2{}_{tr}) = V_{tr}$$

$$1/2\,(V^1{}_{tz} + V^2{}_{tz}) = V_{tz}$$

Then the equations (1) to (6) become:

$$(1a)\quad X_o + \Delta V_{zx} + \cos\frac{\Phi}{2} \cdot \Delta N_t - 2\sin\frac{\Phi}{2} \cdot V_{tr} = 0$$

$$(2a)\quad Y_o + \Delta V_{zy} + \cos\frac{\Phi}{2} \cdot \Delta V_{tr} + 2\sin\frac{\Phi}{2} \cdot N_t = 0$$

$$(3a)\quad Z_o + \Delta N_z + \Delta V_{tz} = 0$$

$$(4a)\quad M_{xo} + \Delta M_{zx} + d.V_{zy} + a.N_z + \cos\frac{\Phi}{2} \cdot \Delta M_{tt} - 2\sin\frac{\Phi}{2} \cdot M_{tr} - c.\Delta V_{tz} = 0$$

$$(5a)\quad M_{yo} + \Delta M_{zy} - d.V_{zx} + \cos\frac{\Phi}{2} \cdot \Delta M_{tr} + 2\sin\frac{\Phi}{2} \cdot M_{tt} + b.V_{tz} = 0$$

$$(6a)\quad M_{zo} + \Delta M_{zz} - a.V_{zx} + \Delta M_{tz} + c \cdot \cos\frac{\Phi}{2} \cdot \Delta N_t - \frac{b}{2} \cdot \sin\frac{\Phi}{2} \cdot \Delta N_t - b \cdot \cos\frac{\Phi}{2} \cdot V_{tr} - 2c \cdot \sin\frac{\Phi}{2} \cdot V_{tr} = 0$$

In the equations (1a) to (6a) only the average values and the differences of internal forces or moments acting on their respective sections are considered.

The conclusions which can be drawn in considering an arch element having infinitesimal dimensions in the vertical (d) and in the tangential (b) direction, are as follows. The general resultant of the stresses on a horizontal or radial section has an infinitesimal distance from a line parallel to Y axis going through the gravity center of sections. Thus, the moments M_{zy} and M_{tr} can be neglected when compared with other internal moments. Also, the twist moments M_{zz} and M_{xx} are the results of tangential and vertical shear components, respectively, because the radial shear components have infinitesimal arms related to the gravity center of the sections.

As the external forces are proportional to the vertical and tangential dimensions of the arch element and the internal forces are proportional to just one of these dimensions, the external forces and their moments when compared with internal forces and their moments can be disregarded.

For an infinitesimal arch element in equation (5a), $M_{yo} = 0$. Similarly,

$$M_{zy} = M_{tr} = 0, \cos\frac{\Phi}{2} = 1, \sin\frac{\Phi}{2} = 0;$$

then

$$dV_{zx} = bV_{tz} \text{ or } \frac{V_{zx}}{b} = \frac{V_{tz}}{d} .$$

Thus, the tangential shear per unit width is equal to the vertical shear per unit height on the same point of the dam.

As the twist moments M_{zz} and M_{tt} are related only to tangential and vertical stresses, respectively, and these stresses are equal on the several portions of the infinitesimal arch element,

$$\frac{M_{zz}}{b} = \frac{M_{tt}}{d} ;$$

that is, the vertical twist moment per unit width is equal to the horizontal twist moment per unit height on the same point of the dam.

In addition to the six equilibrium conditions (equations (1a) to (6a)), the forces and moments acting on the arch element should satisfy the continuity conditions; that is, the deformations of the arch element resulting from these forces and moments must be in agreement with the deformations of the adjacent elements caused by their own forces and moments. As a special case of continuity conditions we have boundary conditions which state that the deformations of the arch elements must agree with those of the adjacent boundaries; that is, the deformations of the arch elements must agree with those of the adjacent abutments.

The relative deformation of two opposite sections of an arch element may be resolved in three translations along the coordinate axis and three rotations around these axes. These six displacements lead to six independent continuity conditions. The six continuity conditions with the six equilibrium conditions are necessary and sufficient to determine the 12 unknown internal forces and moments.

If the normal force N_z, the bending moments M_{zx}, and M_{zy} on a horizontal section of an arch element are known, and a plane distribution of normal stresses is assumed, all normal stresses on this section can be obtained. In

the same way, if N_t, M_{tz}, and M_{tr} are known and a plane distribution of normal stresses on radial planes is assumed, all normal stresses on this section of the arch element can be found.

If the shears V_{zx} and V_{zy} and the twisting moment M_{zz} on a horizontal section of an arch element are known, and uniform distribution of radial shear stress and linear distribution for the tangential shear stresses are assumed, all shear stresses on this section can be obtained. Similarly, if V_{tr}, V_{tz}, and M_{tt} are known and uniform distribution for radial shear stresses and linear distribution for the vertical shear stresses are assumed, all stresses on radial sections of the arch element can be resolved.

One means of translating the continuity conditions in mathematical form consists in relating the deformations of the dam to deformations of horizontal arches and vertical cantilevers. For this purpose the equilibrium conditions may be changed, adding and subtracting in each of the equations (1a) to (6a) an external force (X, Y, Z) or moment (M_x, M_y, M_z). These fictitious forces and moments should be of similar nature as those included in the equations as real external loads. With this introduction and a convenient grouping of the terms, equations (1a) to (6a) become:

$$(7)\quad (X_o - X + \Delta V_{zx}) + (X + \cos\frac{\phi}{2} \cdot \Delta N_t - 2\sin\frac{\phi}{2} \cdot V_{tr}) = 0$$

$$(8)\quad (Y_o - Y + \Delta V_{zy}) + (Y + \cos\frac{\phi}{2} \cdot \Delta V_{tr} + 2\sin\frac{\phi}{2} \cdot N_t) = 0$$

$$(9)\quad (Z_o - Z + \Delta N_z) + (Z + \Delta V_{tz}) = 0$$

$$(10)\quad (M_{xo} - M_x + \Delta M_{zx} + d.V_{zy} + a.N_z) + (M_x + \cos\frac{\phi}{2} \cdot \Delta M_{tt}$$
$$- 2\sin\frac{\phi}{2} \cdot M_{tr} - c.\Delta V_{tz}) = 0$$

$$(11)\quad (M_{yo} - M_y + \Delta M_{zy} - d.V_{zx}) + (M_y + \cos\frac{\phi}{2} \cdot \Delta M_{tr}$$
$$+ 2\sin\frac{\phi}{2} \cdot M_{tt} + b.V_{tz}) = 0$$

$$(12)\quad (M_{zo} - M_z + \Delta M_{zz} - a.V_{zx}) + (M_z + \Delta M_{tz} + c \cdot \cos\frac{\phi}{2} \cdot \Delta N_t$$
$$- \frac{b}{2} \cdot \sin\frac{\phi}{2} \cdot \Delta N_t - b \cdot \cos\frac{\phi}{2} \cdot V_{tr}$$
$$- 2c \cdot \sin\frac{\phi}{2} \cdot V_{tr}) = 0$$

For some values of the additional loads X, Y, Z, M_x, M_y, and M_z, the first part of the equations between parentheses becomes equal to zero. Therefore, for these values of the additional loads, the second part of the equations must equal zero. Then the parts of the equations can be grouped as follows:

$$(8a) \quad (Y_o - Y) + \Delta V_{zy} = 0$$

$$(9a) \quad (Z_o - Z) + \Delta N_z = 0$$

$$(10a) \quad (M_{xo} - M_x) + \Delta M_{zx} + d.V_{zy} + a.N_z = 0$$

$$(7b) \quad X + \cos\frac{\phi}{2} . \Delta N_t - 2\sin\frac{\phi}{2} . V_{tr} = 0$$

$$(8b) \quad Y + \cos\frac{\phi}{2} . \Delta V_{tr} + 2\sin\frac{\phi}{2} . N_t = 0$$

$$(12b) \quad M_z + \Delta M_{tz} + c . \cos\frac{\phi}{2} . \Delta N_t - \frac{b}{2} . \sin\frac{\phi}{2} . \Delta N_t$$

$$- b . \cos\frac{\phi}{2} . V_{tr} - 2c . \sin\frac{\phi}{2} . V_{tr} = 0$$

$$(7a) \quad (X_o - X) + \Delta V_{zx} = 0$$

$$(11a) \quad (M_{yo} - M_y) + \Delta M_{zy} - d.V_{zx} = 0$$

$$(12a) \quad (M_{zo} - M_z) + \Delta M_{zz} - a.V_{zx} = 0$$

$$(9b) \quad Z + \Delta V_{tz} = 0$$

$$(10b) \quad M_x + \cos\frac{\phi}{2} . \Delta M_{tt} - 2\sin\frac{\phi}{2} . M_{tr} - c.\Delta V_{tz} = 0$$

$$(11b) \quad M_y + \cos\frac{\phi}{2} . \Delta M_{tr} + 2\sin\frac{\phi}{2} . M_{tt} + b.V_{tz} = 0$$

The system (8a), (9a), (10a) gives the equations for cantilevers loaded by $(Y_o - Y)$, $(Z_o - Z)$ and $(M_{xo} - M_x)$, which bend in radial planes, being subjected to displacements in radial and vertical directions and rotations around the X axis.

The system (7b), (8b), (12b) gives the equations for arches loaded by X, Y and M_z, which bend in horizontal planes, being subjected to displacements in radial and tangential directions and rotation around the Z axis.

The system (7a), (11a), (12a) gives the equations for cantilevers loaded by $(X_o - X)$, $(M_{yo} - M_y)$ and $(M_{zo} - M_z)$ which bend in transversal planes and twist, being subjected to displacements in tangential direction and rotations around the Y and Z axes.

Similarly, the system (9b), (10b), (11b) gives the equations for arches loaded by Z, M_x and M_y, which bend in vertical direction and twist, being subjected to displacements in vertical direction and rotations around the X and Y axes.

It should be noted that each translation or rotation appears simultaneously on cantilevers radially or transversely loaded and on arches radially or vertically loaded.

The additional loads X, Y, Z, M_x, M_y and M_z act in the same way as the external loads X_o, Y_o, Z_o, M_{xo}, M_{yo} and M_{zo} on the center of the arch element or on the axis of the arches and cantilevers which intercept in the space defined by this arch element. Because of the symmetrical action of the additional loads on the arches and cantilevers, they are usually referred to as "self-balancing loads."

With the use of the self-balancing loads as auxiliary unknowns the equilibrium conditions of the arch element can be sub-divided into four independent systems which express the equilibrium conditions of vertical cantilevers and horizontal arches. In this analysis the use of horizontal arches and vertical cantilevers as principal structures for the calculation of stresses in arch dams is justified, because it is possible to separate the internal forces and moments of the arch element, relating each group to the corresponding structure.

The deformations of the cantilevers and arches are functions of the internal forces and moments. These forces and moments are related to the self-balancing loads X, Y, Z, M_x and M_z by the last four systems of equations. The equality of deformations of the arches and cantilevers on each point of the dam gives six conditions which determine the values of the six unknown self-balancing loads on these points.

As the equations relating the deformations directly to the self-balancing loads are too intricate for a direct solution of the problem, the trial load method was developed. In this method the value of the self-balancing loads is estimated and the loads are then applied with the real external loads as shown in the explanation of the four systems of equations. Then the internal forces and moments are computed, as well as the corresponding deformations of arches and cantilevers by the usual methods of calculation for this kind of structure. This trial estimate of the self-balancing loads is repeated until a reasonable agreement between deformations of arches and cantilevers is reached. Each estimate is oriented by the results of the former trials, leading to a convergence of the results. The computation is usually performed for a few representative arches of unit height and a few cantilevers of unit width on the axis of the dam.

As each external and self-balancing load causes predominantly one kind of displacement or rotation, it is customary to apply these loads successively to the structure elements. This is generally performed in the following sequence.

The first set of loads consists of radial forces (Y_o - Y) on the cantilevers and Y on the arches. With these loads the cantilevers displace in radial and vertical directions and rotate about the X axis; the arches deflect in radial and tangential directions and rotate about the Z axis. The value of these loads is estimated by trial until a reasonable agreement is reached between the radial deformations of arches and cantilevers. These are the predominant movements resulting from these loads.

The second set of loads consists of tangential forces (X_o - X) on the cantilevers and X on the arches. With these loads the arches and cantilevers

deform principally in the tangential direction. These loads are chosen by trial to obviate the discrepancy of tangential deformations on arches and cantilevers caused by other sets of loads and primarily by the first one. In this way, the tangential displacements of the arches caused by the first and second set of loads are added and equal to the tangential displacements of the cantilevers due to the forces X. The tangential loads cause also rotations around Y and Z axes on the cantilevers, as well as radial displacements and rotations around Z axis on the arches. The radial displacements of the arches are added to similar deformations caused by the first set of loads. If this results in a large discrepancy of radial displacements of arches and cantilevers, the first set of loads is readjusted to accomplish this condition to a reasonable degree.

The third set of loads consists of vertical twist moments (M_{zo} - M_z) on the cantilevers and M_z on the arches. With these loads the cantilevers and arches rotate mostly around Z axis. The rotations of the arches are added to similar deformations caused by the first set of loads. There may be also added, respectively, the rotations around the Z axis on the cantilevers and arches due to the second set of tangential loads, which are generally of less magnitude. With this set of loads a reasonable agreement may be reached between rotations around the Z axis on corresponding points of arches and cantilevers. The twist loads also cause tangential displacements and rotations around the Y axis on the cantilevers, as well as radial and tangential displacements on the arches. The tangential displacements of the cantilevers and arches are added to similar deformations due to the first and second set of loads and, if necessary, the tangential trial loads readjusted for agreement of tangential deformations of arches and cantilevers. The radial displacements of the arches are added to similar deformations due to the first set of loads and, if necessary, the radial trial loads readjusted for agreement of radial deformations of arches and cantilevers.

Sometimes the two radial readjustments due to changes introduced by the second and the third set of loads are performed simultaneously, adding radial displacements caused by all loads. Each new adjustment may change those previously performed, but the results are quickly convergent.

These three sets of loads lead to the often-called radial, tangential, and twist adjustments. The other sets of loads and corresponding adjustments are considered in special and simplified manner, as will be explained.

A set of loads consisting of horizontal twist moments (M_{xo} - M_x) on the cantilevers and M_x on the arches may now be applied. These loads should eliminate the disagreement in rotations of arches and cantilevers around X axis caused by the first set of loads. However, the values of the internal moments M_{zz} from the third set of loads are known with sufficient accuracy by integration of the self-balancing twists (M_{zo} - M_z) along the cantilevers (equation 12a). As the horizontal and vertical twist moments are equal on the same point ($\frac{M_{tt}}{d} = \frac{M_{zz}}{b}$), the distribution of these moments along the dam are known and the values of the loads M_x may be obtained by differentiation along arches (equation 10b). These values of the self-balancing loads (M_{xo} - M_x) and M_x lead automatically to an agreement in rotations of cantilevers and arches around the X axis. In this way, it is not necessary to make the horizontal twist adjustment. In the radial adjustment the changes in radial deflections of the cantilevers caused by the horizontal twist loads (M_{xo} - M_x) and M_x should be considered. The self-balancing twist loads cause also vertical displacements on the cantilevers and arches as well as rotations around the Y axis on the arches.

The next set of loads to be applied consists of vertical forces $(Z_o - Z)$ on the cantilevers and Z on the arches. These loads should eliminate the disagreement in vertical displacements of cantilevers and arches caused predominantly by the horizontal self-balancing twist loads. The vertical displacements of the cantilevers due to the horizontal twist and radial loads appear because of the angle between the cantilever axis and the vertical, and may be neglected in most cases. The vertical displacements of the arches due to the vertical bending caused by the horizontal twist loads are generally of more importance. Known with sufficient accuracy are the values of the internal forces V_{zx} from the second set of self-balancing loads. These values were obtained by integration of the forces $(X_o - X)$ along the cantilevers (equation 7a). As the tangential and vertical shear forces are equal on the same point $(\frac{V_{tz}}{d} = \frac{V_{zx}}{b})$, the distribution of these shears along the dam are known and the self-balancing loads Z may be obtained by differentiation along arches (equation 9b). These values of the self-balancing loads $(Z_o - Z)$ and Z lead automatically to an agreement in vertical displacements of cantilevers and arches. In this way, it is not necessary to make the vertical adjustment. The vertical loads $(X_o - X)$ affect the radial deflections of the cantilevers to some extent, and these loads should be considered in the radial adjustment. The vertical self-balancing loads also cause rotations around the X axis on the cantilevers as well as rotations around the X and Y axes on the arches; these rotations are generally of minor effect.

Knowing the values of the horizontal and vertical self-balancing loads and consequently those of the internal shears V_{zx} and V_{tz}, the values of the twist moments M_{zz}, M_{tt}, and M_x, according to equations 12a and 10b, may be more accurately determined.

It should be noted that the vertical self-balancing loads cause longitudinal deformations on the cantilevers and transversal bending on the arches. A study of these deformations shows that the vertical self-balancing loads cause relatively small vertical displacements. As the vertical displacements caused by other loads are also negligible, the horizontal arches of the dam deform virtually in their own planes.

There is left one set of self-balancing loads which consist of radial moments $(M_{yo} - M_y)$ on the cantilevers and M_y on the arches. These loads eliminate the disagreement in rotation of arches and cantilevers around the Y axis, which is caused by tangential and vertical loads as well as by vertical and horizontal twists. This condition may be accomplished in a simplified manner with the aid of the former stated conclusion that the horizontal arches deform without appreciable vertical displacements. This shows that the cantilevers deform transversely keeping their sections nearly horizontal, being subjected to shear detrusion and almost no bending deformation. This means that in each element of the cantilevers the moment $(M_{yo} - M_y)$ equals approximately the moment $d.V_{zx}$ of the tangential shears; equation 11a thus reduces to $\Delta M_{zy} \cong 0$. Thus we have a simple way to consider the self-balancing load $(M_{yo} - M_y)$ and M_y, stating that the transversal displacements of the cantilevers are produced almost entirely by the shear detrusion of $\Delta V_{zx} = X - X_o$ given by equation 7a. In the arches the moments M_y have the effect of virtually eliminating the vertical displacements caused by the shear V_{tz}. The radial moments $(M_{yo} - M_y)$ and M_y also cause rotations around Z axis on the cantilevers and rotations around X axis on the arches, which are usually negligible.

Thus, in reviewing all effects of the self-balancing and external loads, it is

seen that the three first sets of loads with radial, tangential, and twist adjustments give a reasonably accurate result for the internal stresses of a dam, particularly when they are followed by the corresponding readjustments.

In regard to the other loads, consideration should be given to the radial deflections caused by horizontal twists ($M_{xo} - M_x$) on the cantilevers and the fact that tangential deformations of cantilevers result only from the shear detrusion due to $\Delta V_{zx} = X - X_o$. The other secondary influences of vertical loads, horizontal twists, and radial moments can be usually neglected but should be kept in mind for special structure conformations.

The self-balancing loads may also be applied simultaneously as shown in the last four systems of equations including all primary and secondary effects at the same time. For small vertical displacements, and disregarding the secondary effects of vertical loads and horizontal twists on the deformations of the cantilevers, the equations simplify to:

$$(13)\qquad \begin{aligned} &(Y_o - Y) + \Delta V_{zy} = 0 \\ &\Delta M_{zx} + dV_{zy} = 0 \end{aligned}$$

$$(14)\qquad \begin{aligned} &X + \cos\frac{\phi}{2} \cdot \Delta N_t - 2\sin\frac{\phi}{2} \cdot V_{tr} = 0 \\ &Y + \cos\frac{\phi}{2} \cdot \Delta V_{tr} + 2\sin\frac{\phi}{2} \cdot N_t = 0 \\ &M_z + \Delta M_{tz} + c \cdot \cos\frac{\phi}{2} \cdot \Delta N_t - \frac{b}{2} \cdot \sin\frac{\phi}{2} \cdot \Delta N_t \\ &\qquad - b \cdot \cos\frac{\phi}{2} \cdot V_{tr} - 2c \cdot \sin\frac{\phi}{2} \cdot V_{tr} = 0 \end{aligned}$$

$$(15)\qquad \begin{aligned} &(X_o - X) + \Delta V_{zx} = 0 \\ &\Delta M_{zy} = 0 \\ &(M_{zo} - M_z) + \Delta M_{zz} - a . V_{zx} = 0 \end{aligned}$$

The last of the four systems does not appear because the agreement in vertical displacements and rotation around the X axis are automatically satisfied if the tangential and twist adjustments are taken into consideration.

In the previous discussion Poisson's ratio was different from zero ($\mu \neq 0$) for shear deformations, but the transversal deformations due to longitudinal and flexure loads were not considered. Even with the separation of the internal forces for cantilevers and arches gained with the introduction of the self-balancing loads, the deformations of these structure elements are not independent when transversal deformations are considered. In this case the

following relations exist between internal forces or moments and corresponding deformations:

$$N_z - \mu N_t = \text{longitudinal deformation of cantilevers}$$

$$N_t - \mu N_z = \text{longitudinal deformation of arches}$$

$$M_{zx} + \mu M_{tz} = \text{flexure deformation of cantilevers}$$

$$M_{tz} + \mu M_{zx} = \text{flexure deformation of arches}$$

The action of water loads on the upstream face of the dam causes a variable radial compression and a consequent small normal and flexure deformation on arches and cantilevers.

This influence of transversal deformations can be considered in the computation of stresses in dams by a convenient adjustment of the trial loads, which is not usually important.

If the arch dam has multiple centers the cantilevers formed by superposition of elements with radial faces will have a warped shape. This causes an interference of the radial and transversal components of the cantilever deformations. When the cantilevers are not excessively warped, they can be replaced without appreciable error, by straight cantilevers having vertical faces passing through an intermediate center of the arches.

The joints of an arch dam are usually grouted after the monoliths are constructed to their full height. In this case the weight of the concrete is carried only by the cantilevers and is not considered as load in this analysis.

The Use of Self-Balancing Loads in the Analysis of Straight Gravity Dams

In the computation of stresses for large straight gravity dams consideration should be given to the horizontal transfer of loads. This is particularly true where the canyon is narrow or abutments are steep.

The deductions developed for arch dams may be applied to straight gravity dams. Then,

$$\phi = 0, \ \sin\frac{\phi}{2} = 0, \ \cos\frac{\phi}{2} = 0, \ c = 0,$$

and the horizontal arches are replaced by straight beams. For simplicity, the direction parallel to the Y axis is still designated as radial.

The radial and both twist loads cause no vertical and transversal displacements on the beams and only small deformations of this kind on the cantilevers. The tangential and vertical self-balancing loads needed for the agreement of cantilever and beam deformations are consequently small.

The longitudinal rigidity of beams and cantilevers prevent the dam from deforming appreciably in the vertical and tangential directions. Consequently, the rotations around the Y axis are small and the value of the self-balancing loads M_y can be neglected. Consequently, the following self-balancing loads can be disregarded:

$$X = Z = M_y = 0,$$

and for the external loads:

$$X_o = Z_o = M_{yo} = 0.$$

Then equations 9a, 7b, 7a, 11a, 9b, 11b become:

$$N_z = V_{zx} = M_{zy} = 0, \; N_t = V_{tz} = M_{tr} = 0.$$

The weight of the concrete, vertical earthquake forces, and vertical water load are transferred directly to the cantilevers. The tangential earthquake forces are taken directly by the horizontal beams.

With these simplifications the four systems of equations established for arch dams become:

$$(16) \qquad \begin{aligned} &(Y_o - Y) + \Delta V_{zy} = 0 \\ &(M_{xo} - M_x) + \Delta M_{zx} + d.V_{zy} = 0 \end{aligned}$$

$$(17) \qquad \begin{aligned} &Y + \Delta V_{tr} = 0 \\ &M_z + \Delta M_{tz} - b.V_{tr} = 0 \end{aligned}$$

$$(18) \qquad (M_{zo} - M_z) + \Delta M_{zz} = 0$$

$$(19) \qquad M_x + \Delta M_{tt} = 0$$

The system (16) gives the equations for cantilevers loaded by $(Y_o - Y)$ and $(M_{xo} - M_x)$, which bend in radial planes, being subjected to displacements in radial direction and rotations around the X axis.

The system (17) gives the equations for beams loaded by Y and M_z, which bend in horizontal planes, being subjected to displacements in radial direction and rotations around the Z axis.

The equation (18) defines cantilevers loaded by $(M_{zo} - M_z)$, twisting vertically, being subjected to rotations around the Z axis.

The equation (19) defines beams loaded by M_x, twisting horizontally, being subjected to rotations around the X axis.

Instead of introducing all self-balancing loads at the same time, they may be applied successively according to the importance of their effects. This is generally performed in the following sequence:

The first set of loads consists of radial forces $(Y_o - Y)$ on the cantilevers and Y on the beams. With these loads the cantilevers displace in radial direction and rotate around the X axis, and the beams deflect in radial direction

and rotate around the Z axis. The value of these loads is estimated by trial until a reasonable agreement is reached between radial deformations of beams and cantilevers.

The second set of loads consists of vertical twist moments (M_{zo} - M_z) on the cantilevers and M_z on the beams. With these loads the cantilevers and beams rotate around the Z axis. With this set of loads there is eliminated the disagreement of rotations around the Z axis on cantilevers and beams due to the first set of loads. The twist loads also cause radial deformations in the beams. These radial displacements of the beams are added to similar deformations due to the first set of loads and, if necessary, these loads are readjusted to reach an agreement in radial deformations of beams and cantilevers.

These two sets of loads lead to the often-called radial and twist adjustments.

Next, application can be made of a set of loads consisting of horizontal twist moments (M_{xo} - M_x) on the cantilevers and M_x on the beams. These loads should eliminate the disagreement in rotations around the X axis on cantilevers and beams due to the first set of loads. However, known with sufficient accuracy are the values of the internal moments M_{zz} from the second set of loads, obtained by integration of the self-balancing twists (M_{zo} - M_z) along the cantilevers (equation 18). As the horizontal and vertical twist moments are equal on the same point ($\frac{M_{tt}}{d} = \frac{M_{zz}}{b}$), distribution of these moments along the dam is known and the values of the loads M_x may be determined by differentiation along beams (equation 19). These values of the self-balancing loads (M_{xo} - M_x) and M_x lead automatically to an agreement in rotations of cantilevers and beams around the X axis. In this way, it is not necessary to make the horizontal twist adjustment. In the radial adjustment consideration should be given to the changes in radial deflections of the cantilevers caused by the horizontal twist loads (M_{xo} - M_x).

Sometimes, to eliminate high flexure stresses on the horizontal beams of straight gravity dams, the vertical joints are not grouted. In this case the internal moments M_{tz} and M_{tt} disappear in these joints; remaining are the shears V_{tr} when there are vertical keyways between monoliths. The corresponding equations simplify to:

$$(20)\qquad \begin{aligned} &(Y_o - Y) + \Delta V_{zy} = 0 \\ &M_{xo} + \Delta M_{zx} + d.V_{zy} = 0 \end{aligned}$$

$$(21)\qquad \begin{aligned} &Y + \Delta V_{tr} = 0 \\ &M_z - b.V_{tr} = 0 \end{aligned}$$

$$(22)\qquad (M_{zo} - M_z) + \Delta M_{zz} = 0$$

According to equation (19) $M_{tt} = M_x = O$ and M_x disappears in the system (16).

Analyzing the system (21) it is seen that the radial loads Y are related to the torsions M_z and consequently to $(M_{zo} - M_z)$. This shows that the beams (equation 21) and twisted cantilevers (equation 22), have no more independent loads and cannot be treated as separate structures.

Substituting M_z from equation (22) in equation (21) these systems become:

$$(20) \quad \begin{aligned} (Y_o - Y) + \Delta V_{zy} &= 0 \\ M_{xo} + \Delta M_{zx} + d.V_{zy} &= 0 \end{aligned}$$

$$(21a) \quad \begin{aligned} Y + \Delta V_{tr} &= 0 \\ M_{zo} + \Delta M_{zz} - b.V_{tr} &= 0 \end{aligned}$$

The system (20) gives the equations for cantilevers loaded by $(Y_o - Y)$, Y, and M_{xo}, which bend in radial planes, being subjected to radial displacements and rotations around the X axis.

The system (21a) gives the equations for a structure loaded by Y and M_{zo}, resisting to shear in horizontal planes and to twist in vertical directions. The differences of twist moments in horizontal sections balance the moments of the radial shears V_{tr} in vertical sections and the differences of these shears give the radial loads Y carried by this structure. The angular deformations are due to torsions of cantilevers. The radial deflections are equal to the sum of shear detrusion in a horizontal plane and the integral of angular deformations along this plane. This structure is designated in this paper as the horizontal twist structure.

Thus in straight gravity dams with ungrouted joints there are two principal structures, the cantilevers and the twisted structure. The self-balancing loads are the radial forces $(Y_o - Y)$ on the cantilevers and Y on the twisted structure. These loads are estimated by trial until an agreement is made between the radial deflections of cantilevers and all points of the twisted structure.

In contrast to the cantilevers, which may be in any number independent one from the other, the twisted structure is just one occupying the whole dam.

A single load on the twisted structure produces shears on the horizontal plane passing through the load. These shears are constant on each side of the application point. The shears are balanced by vertical twist moments which extend between the horizontal plane passing through the load and the fixed boundaries being constant on each side of the load. The structure is one time statically indeterminate and the value of the shear force on one side of the load, for instance, can be determined from radial deformation conditions at any point. In a symmetrical structure with symmetrical loads the shears disappear in the line of symmetry and the system becomes statically determinate.

The Use of Twisted Structures in the Analysis of Arch Dams

The use of twisted structures presents another method of relating the stress distribution of arch dams to principal systems of simple calculation. In this method the radial shears on horizontal and vertical sections are subdivided in two parts, one equilibrating flexure moments and the other equilibrating twist moments. The external loads are supported by four kinds of structures: vertical cantilevers, horizontal arches, horizontal twisted structures, and vertical twisted structures. To show which loads should be related to each kind of structure and to justify mathematically the grouping of the internal forces and moments for the equilibrium conditions of these structures, the external loads X_o, Y_o, Z_o, M_{yo} and the internal radial shears V_{zy}, V_{tr} are divided as follows:

$$(23)\quad \begin{aligned} X_o &= X^a + X^b \\ Y_o &= Y^a + Y^b + Y^c + Y^d \\ Z_o &= Z^a + Z^b \\ M_{yo} &= M^a{}_y + M^b{}_y \\ V_{zy} &= V^a{}_{zy} + V^b{}_{zy} \\ V_{tr} &= V^a{}_{tr} + V^b{}_{tr} \end{aligned}$$

The six equilibrium conditions (equations 1a to 6a) with the terms conveniently grouped become:

$$(24)\quad (X^a + \Delta V_{zx} - 2 \sin \frac{\phi}{2} \cdot V^a{}_{tr}) + (X^b + \cos \frac{\phi}{2} \cdot \Delta N_t - 2 \sin \frac{\phi}{2} \cdot V^b{}_{tr}) = 0$$

$$(25)\quad (Y^a + \Delta V^a{}_{zy}) + (Y^b + \cos \frac{\phi}{2} \cdot \Delta V^b{}_{tr} + 2 \sin \frac{\phi}{2} \cdot N_t) + (Y^c + \cos \frac{\phi}{2} \cdot \Delta V^a{}_{tr}) + (Y^d + \Delta V^b{}_{zy}) = 0$$

$$(26)\quad (Z^a + \Delta N_z) + (Z^b + \Delta V_{tz}) = 0$$

$$(27)\quad (M_{xo} + \Delta M_{zx} + d.V^a{}_{zy} + a.N_z) + (d.V^b{}_{zy} + \cos \frac{\phi}{2} \cdot \Delta M_{tt} - 2 \sin \frac{\phi}{2} \cdot M_{tr} - c.\Delta V_{tz}) = 0$$

(28) $(M^a{}_y + \Delta M_{zy} - d.V_{zx}) + (M^b{}_y + \cos\frac{\Phi}{2} . \Delta M_{tr} + 2\sin\frac{\Phi}{2} . M_{tt}$

$+ b.V_{tz}) = 0$

(29) $(\Delta M_{zz} - a.V_{zx} - b . \cos\frac{\Phi}{2} . V^a{}_{tr} - 2c . \sin\frac{\Phi}{2} . V^a{}_{tr})$

$+ (M_{zo} + \Delta M_{tz} + c . \cos\frac{\Phi}{2} . \Delta N_t - \frac{b}{2} . \sin\frac{\Phi}{2} . \Delta N_t$

$- b . \cos\frac{\Phi}{2} . V^b{}_{tr} - 2c . \sin\frac{\Phi}{2} . V^b{}_{tr}) = 0$

For some values of the parts of the loads X_o, Y_o, Z_o, M_{yo} and internal forces V_{zy}, V_{tr}, the portions of these equations between parentheses become equal to zero. In this case the parts of the equations can be grouped as follows:

(25a) $Y^a + \Delta V^a{}_{zy} = 0$

(26a) $Z^a + \Delta N_z = 0$

(27a) $M_{xo} + \Delta M_{zx} + d.V^a{}_{zy} + a.N_z = 0$

(24b) $X^b + \cos\frac{\Phi}{2} . \Delta N_t - 2\sin\frac{\Phi}{2} . V^b{}_{tr} = 0$

(25b) $Y^b + \cos\frac{\Phi}{2} . \Delta V^b{}_{tr} + 2\sin\frac{\Phi}{2} . N_t = 0$

(29b) $M_{zo} + \Delta M_{tz} + c . \cos\frac{\Phi}{2} . \Delta N_t - \frac{b}{2} . \sin\frac{\Phi}{2} . \Delta N_t$

$- b . \cos\frac{\Phi}{2} . V^b{}_{tr} - 2c . \sin\frac{\Phi}{2} . V^b{}_{tr} = 0$

(25c) $Y^c + \cos\frac{\Phi}{2} . \Delta V^a{}_{tr} = 0$

(24a) $X^a + \Delta V_{zx} - 2\sin\frac{\Phi}{2} . V^a{}_{tr} = 0$

(29a) $\Delta M_{zz} - a.V_{zx} - b . \cos\frac{\Phi}{2} . V^a{}_{tr}$

$- 2c . \sin\frac{\Phi}{2} . V^a{}_{tr} = 0$

(28a) $$M^a_y + \Delta M_{zy} - d.V_{zx} = 0$$

(25d) $$Y^d + \Delta V^b_{zy} = 0$$

(26b) $$Z^b + \Delta V_{tz} = 0$$

(27b) $$d.V^b_{zy} + \cos\frac{\phi}{2} \cdot \Delta M_{tt} - 2\sin\frac{\phi}{2} \cdot M_{tr} - c.\Delta V_{tz} = 0$$

(28b) $$M^b_y + \cos\frac{\phi}{2} \cdot \Delta M_{tr} + 2\sin\frac{\phi}{2} \cdot M_{tt} + b.V_{tz} = 0$$

The system (25a), (26a), (27a) gives the equations for cantilevers loaded by Y^a, Z^a and M_{XO}, bending in radial planes, being subjected to displacements in radial and vertical directions and rotations around the X axis.

The system (24b), (25b), (29b) gives the equations for arches loaded by X^b, Y^b and M_{ZO}, bending in horizontal planes, being subjected to displacements in radial and tangential directions and rotations around the Z axis.

The system (25c), (24a), (29a), (28a) gives the equations for a horizontal twisted structure, loaded by X^a, Y^c and M^a_y, deflecting radially, twisting vertically and bending transversely, being subjected to radial and tangential displacements and rotations around the Y and Z axes.

The system (25d), (26b), (27b), (28b) gives the equations for a vertical twisted structure loaded by Y^d, Z^b and M^b_y, deflecting radially, twisting horizontally and bending vertically, being subjected to radial and vertical displacements and rotations around the X and Y axes.

In the four structures there are three independent conditions of equality in radial deformations and five more conditions of equal deformations along the X and Z axes as well as equal rotations around the three coordinate axes. These eight conditions with the six equations (23) determine the values of the 14 unknown components of loads and internal shears shown on the second member of these equations. With these parts of the external loads and internal shears known, the other internal forces and moments can be determined from the last four systems of equations.

In contrast to the cantilevers and arches which may be in any number independent from the others, there is only one horizontal twisted structure and one vertical twisted structure, each occupying the whole dam.

In the horizontal twisted structure the rotations around the Z axis are due to torsions in the vertical elements of the structure. The radial displacements are equal to the sum of shear detrusions in the horizontal plane passing through the considered point and to the integral of the rotations around the Z axis along this plane caused by the torsion of the vertical elements. The tangential displacements and rotations around the Y axis due to X^a, M^a_y and the tangential components of the shears V_{tr} are calculated from transversal bending of vertical elements in the structure.

With an isolated radial load on one point of the horizontal twisted structure there are radial shears V_{tr} only on the horizontal plane passing through the load, which are constant on each side of this load.

These shears are balanced mostly by vertical twist moments M_{zz} which

act between the horizontal plane passing through the load and the foundations, being constant on each side of the load. The discontinuity of the radial shears V_{tr} on the loaded point causes a tangential component which bends the vertical elements passing through the load, resulting in tangential displacements and rotations around the Y axis in this vertical element. Tangential loads and M^a_y moments cause transversal bending of the vertical elements of the structure, with tangential displacements and rotations around the Y axis, similar to transversely loaded cantilevers.

The horizontal twisted structure is then one time statically indeterminate for an isolated radial load. If the structure and the loads are symmetrical, the radial shears disappear on the plane of symmetry and the other internal forces are statically determinate.

In the vertical twisted structure the rotations around the X axis are due to torsions of horizontal elements in the structure. The radial displacements are equal to the sum of shear detrusions in the vertical plane passing through the load and the integral of the rotations around the X axis along this plane caused by torsion of horizontal elements. The vertical displacements and rotation around the Y axis due to Z^b, M^b_y and radial components of the twists M_{tt} are calculated from vertical bending of horizontal elements in the structure.

With an isolated radial load on one point of the vertical twisted structure, we have radial shears V_{zy} only on the vertical planes passing through the load; the shears are constant between the load and the foundation. These shears are balanced mostly by horizontal twist moments M_{tt} and radial flexure moments M_{tr} which act between the horizontal plane passing through the load and the abutments. The vertical loads Z^b and M^b_y moments cause vertical bending of the horizontal elements in the structure, with vertical displacements and rotations around the Y axis. These loads Z^b and M^b_y cause also torsion on the horizontal elements, affecting consequently the rotations around the X axis of the twisted structure.

The vertical twisted structure is then one time statically indeterminate for an isolated radial load. For isolated vertical loads or M^b_y moments, the structure is three times statically indeterminate because of the fixed supports on both ends of the horizontal elements. If the structure and the loads are symmetrical, the vertical shears and horizontal twists disappear in the plane of symmetry and there remain only the bending moments M_{tr} as statically indeterminate unknowns in this plane.

It should be noted that the radial loads carried by the horizontal and vertical twisted structures in a dam are equal as will be proved. Writing equations (25c) and (29a) for an arch element with infinitesimal dimensions in vertical and tangential directions, we have:

$$Y^c = \frac{\partial V_{tr}}{\partial X}\, dX \,, \qquad V_{tr} = \frac{\partial M_{zz}}{\partial Z}\, dZ$$

or substituting the second in the first:

$$Y^c = \frac{\partial^2 M_{zz}}{\partial X . \partial Z}\, dX . dZ \;.$$

In the same way, equations (25d) and (27b) for an infinitesimal arch element lead to:

$$Y^d = \frac{\partial V_{zy}}{\partial Z} dZ \quad , \quad V_{zy} = \frac{\partial M_{tt}}{\partial X} dX$$

and consequently:

$$Y^d = \frac{\partial^2 M_{tt}}{\partial Z . \partial X} dZ . dX \; .$$

As the moments M_{zz} per unit width are equal to M_{tt} per unit height and the partial differentiations may be interchanged, there is obtained an equality of radial loads ($Y^c = Y^d$) carried by the two twisted structures.

Instead of applying all loads at the same time, they may be introduced progressively according to the importance of their effects. This may be performed in the following sequence.

First, the four structures are loaded with the four components of the radial loads Y_o estimated by trial until a reasonable agreement is reached between all radial displacements and rotations around the X and Z axes. The radial loads cause also vertical displacements in the cantilevers, tangential displacements in the arches and in the twisted structures, vertical or tangential displacements and rotations around the Y axis. The equality of the radial loads Y^c and Y^d on the twisted structures simplifies the trial estimate of these loads.

The second set of loads consists of tangential forces X^b on the arches and X^a on the horizontal twisted structure, estimating them by trial until there is no disagreement between tangential displacements of these structures due to the first set of loads. The tangential loads also cause radial displacements and rotations around the Z axis in the arches and rotations around the Y axis in the horizontal twisted structure.

Then the vertical loads Z^a and Z^b are applied on the cantilevers and the vertical twisted structure to eliminate the disagreement in vertical displacements of these structures due to the first set of loads. As the vertical and tangential shears per unit length are equal on the same point of the dam, the loads Z^a, Z^b are related to X^a, X^b by the equations (26a), (26b), (24a) and (24b). This results in an automatic agreement of vertical deformations when there is an adjustment in tangential displacements. However, the effect of the vertical loads on the radial displacements and rotations around the X axis of the cantilevers should be considered, as well as rotations around the X and Y axes of the vertical twisted structure.

There is one set of loads left, consisting of moments $M^a{}_y$ and $M^b{}_y$ on the horizontal and vertically twisted structures, respectively, which are estimated by trial to eliminate the disagreement of rotations around the Y axis in these structures due to loads applied in the former adjustments. The small vertical displacements of the dam, as stated when discussing the use of self-balancing loads, show that these loads may be used in a simplified way. Then the moments $M^a{}_y$ become equal to those produced by the shears V_{zx}, and the transversal deformations of the horizontal twisted structure can be calculated as formed of shear detrusions only.

Analyzing the dam in this way requires the use of readjustments due to the change of some deformations caused by the introduction of new sets of loads.

In the case of small vertical displacements and disregarding the secondary effects of vertical loads on the deformations of cantilevers, the equations simplify to:

$$\text{(30)} \quad \begin{cases} Y^a + \Delta V^a_{zy} = 0 \\ M_{xo} + \Delta M_{zx} + d.V^a_{zy} = 0 \end{cases}$$

$$\text{(31)} \quad \begin{cases} X^b + \cos\frac{\phi}{2} \cdot \Delta N_t - 2\sin\frac{\phi}{2} \cdot V^b_{tr} = 0 \\ Y^b + \cos\frac{\phi}{2} \cdot \Delta V^b_{tr} + 2\sin\frac{\phi}{2} \cdot N_t = 0 \\ M_{zo} + \Delta M_{tz} + c \cdot \cos\frac{\phi}{2} \cdot \Delta N_t - \frac{b}{2} \cdot \sin\frac{\phi}{2} \cdot \Delta N_t \\ \quad - b \cdot \cos\frac{\phi}{2} \cdot V^b_{tr} - 2c \cdot \sin\frac{\phi}{2} \cdot V^b_{tr} = 0 \end{cases}$$

$$\text{(32)} \quad \begin{cases} Y^c + \cos\frac{\phi}{2} \cdot \Delta V^a_{tr} = 0 \\ X^a + \Delta V_{zx} - 2\sin\frac{\phi}{2} \cdot V^a_{tr} = 0 \\ \Delta M_{zz} - a.V_{zx} - b \cdot \cos\frac{\phi}{2} \cdot V^a_{tr} - 2c.\sin\frac{\phi}{2}.V^a_{tr} = 0 \\ \Delta M_{zy} = 0 \end{cases}$$

The last of the four systems does not appear because the agreement in vertical displacements is automatically satisfied if we perform the tangential adjustment and the radial loads Y^d are equal to those Y^c of the horizontal twisted structure.

Discussion could also be made here of the effect of transversal deformation which results when Poisson's ratio in compression and flexure is considered. This consideration has generally little effect in the results in calculation of internal stresses in arch dams.

The Use of Twisted Structures in the Analysis of Straight Gravity Dams

The deduction developed for arch dams may be applied to straight gravity dams, setting:

$$\phi = 0, \quad \sin\frac{\phi}{2} = 0, \quad \cos\frac{\phi}{2} = 1, \quad c = 0$$

and replacing the horizontal arches by straight beams.

For simplicity the direction parallel to the Y axis is still designated as radial.

The radial and both twist loads cause no vertical and transversal displacements on the beams and vertical twisted structure, causing mostly small displacements of this kind in the cantilevers and horizontal twisted structure. Consequently, the loads X^a, X^b, Z^a and Z^b which should eliminate the disagreement of vertical and transversal deformations on the four structures are small in most cases.

The longitudinal rigidity of beams and cantilevers prevent the dam from deforming appreciably in vertical and tangential directions. Consequently, the rotations around the Y axis are small and the value of the loads $M^a{}_y$ and $M^b{}_y$ can be neglected.

In this way, the following loads can be disregarded:

$$X^a = X^b = Z^a = Z^b = M^a{}_y = M^b{}_y = 0$$

The equations 26a, 24a, 28a, 24b, 26b, 28b lead then to:

$$N_z = V_{zx} = M_{zy} = 0, \quad N_t = V_{tz} = M_{tr} = 0$$

The weight of the concrete, vertical earthquake forces, and vertical water load are transferred directly to the cantilevers. The tangential earthquake forces are taken directly by the horizontal beams.

With these simplifications the four systems of equations established for arch dams become:

$$(33) \qquad \begin{aligned} & Y^a + \Delta V^a{}_{zy} = 0 \\ & M_{xo} + \Delta M_{zx} + dV^a{}_{zy} = 0 \end{aligned}$$

$$(34) \qquad \begin{aligned} & Y^b + \Delta V^b{}_{tr} = 0 \\ & M_{zo} + \Delta M_{tz} - b.V^b{}_{tr} = 0 \end{aligned}$$

$$(35) \qquad \begin{aligned} & Y^c + \Delta V^a{}_{tr} = 0 \\ & \Delta M_{zz} - b.V^a{}_{tr} = 0 \end{aligned}$$

$$(36) \qquad \begin{aligned} & Y^a + \Delta V^b{}_{zy} = 0 \\ & d.V^b{}_{zy} + \Delta M_{tt} = 0 \end{aligned}$$

The system (33) gives the equations for cantilevers loaded by Y^a and M_{xo}, bending in radial planes, being subjected to radial displacements and rotations around the X axis.

The system (34) gives the equations for beams loaded by Y^b and M_{zo}, bending in horizontal planes, being subjected to radial displacements and rotations around the Z axis.

The system (35) gives the equations for a horizontal twisted structure loaded by Y^c, deflecting radially and twisting vertically, being subjected to radial displacements and rotations around the Z axis.

The system (36) gives the equations for a vertical twisted structure loaded by Y^d, deflecting radially and twisting horizontally, being subjected to radial displacements and rotations around the X axis.

The loads Y^a, Y^b, Y^c and Y^d are estimated by trial until an agreement is reached between radial deflections and rotations around the X and Z axes in the four structures. As the loads Y^c and Y^d carried by the horizontal and vertical twisted structures, respectively, are equal, there is no need to calculate the deformations of the last of these structures.

In the horizontal twisted structure the differences of twist moments M_{zz} are balanced by the moments of radial shears $V^a{}_{tr}$. The differences of these radial shears give the radial load Y^c carried by this structure. In the vertical twisted structure the differences of twist moments M_{tt} are balanced by the moments of radial shears $V^b{}_{zy}$. The differences of these radial shears give the radial loads Y^d carried by the vertical twisted structure.

In the horizontal twisted structure the rotations around the Z axis are due to torsion of the vertical elements. The radial displacements are equal to the sum of shear detrusions in horizontal planes and to the integral of rotations around the Z axis along these planes caused by torsion of vertical elements.

In the vertical twisted structure the rotations around the X axis are due to torsion of vertical elements. The radial displacements are equal to the sum of shear detrusions in vertical planes and to the integral of rotations around the X axis along these planes caused by torsion of horizontal elements.

As in arch dams, the horizontal twisted structure is one time statically indeterminate for an isolated radial load. This is true also for the vertical twisted structure for radial loads. For straight gravity dams there are no transversal and vertical loads on the twisted structures. When the structure and the loads are symmetrical both twisted structures become statically determinate because the shears V_{tr} or the moments M_{tt} disappear in the plane of symmetry.

For straight gravity dams with ungrouted joints but which have still vertical keyways between monoliths, the internal moments M_{tz} and M_{tt} disappear and the equations (33) to (36) become:

$$(37) \quad \begin{cases} Y^a + \Delta V^a{}_{zy} = 0 \\ M_{xo} + \Delta M_{zx} + d.V^a{}_{zy} = 0 \end{cases}$$

$$(38) \quad \begin{cases} Y^b + \Delta V^b{}_{tr} = 0 \\ M_{zo} - b.V^b{}_{tr} = 0 \end{cases}$$

$$(39)\quad \begin{aligned} &Y^c + \Delta V^a_{tr} = 0 \\ &\Delta M_{zz} - b.V^a_{tr} = 0 \end{aligned}$$

As equations (36) disappear because M_{tt} = O and consequently $Y^d = V^b_{zy}$ = O, V^a_{zy} is replaced in equation (37) by V_{zy}. Adding the first and second equations of the systems (38) and (39), respectively, and noting that $V^a_{tr} + V^b_{tr} = V_{tr}$, they may be written as follows:

$$(37a)\quad \begin{aligned} &Y^a + \Delta V_{zy} = 0 \\ &M_{xo} + \Delta M_{zx} + d.V_{zy} = 0 \end{aligned}$$

$$(38a)\quad \begin{aligned} &Y^e + \Delta V_{tr} = 0 \\ &M_{zo} + \Delta M_{zz} - b.V_{tr} = 0 \end{aligned}$$

where:

$$Y^e = Y^b + Y^c = Y_o - Y^a$$

These are the equations defining cantilevers and a horizontal twisted structure similar to equations (20) and (21a) found with the use of self-balancing loads.

There is no real difference between these equations because the use of a self-balancing load Y can be interpreted as a subdivision of the radial load Y_o in two parts as Y^a and Y^d. For straight gravity dams with ungrouted joints the method incorporating self-balancing loads is equal to the method embracing twisted structures.

The parts Y^a and Y^e of the external radial load are estimated by trial until an agreement is made between radial displacements in the cantilevers and the horizontal twisted structure.

The elimination of one of the systems (38) and (39) by adding them is justified because the loads Y^b and M_{zo} are now related by the first of these systems and Y^b is no longer an unknown.

The deformations of the horizontal twisted structure are computed in the same way as explained for straight gravity dams having grouted joints.

CONCLUSION

In this paper methods of analysis have been developed for the most general case of arch dams to the particular cases of straight gravity dams with grouted and ungrouted joints. Usually arch dams were analyzed with the aid of self-balancing loads and straight gravity dams were analyzed using the twisted structure technique. The reason for the distinct use of these two methods is that the self-balancing loads lead to a simpler solution for arch dams and

the use of twisted structures is believed more convenient for straight gravity dams.

The general development of the discussion shows that we may separate the internal forces and moments in the equilibrium conditions of equations (1a) to (6a) in several ways, referring each group to structures of easy calculation. The author considers the possibility of finding an easier method of calculation than those previously used.

Recognizing that the most work in calculating arch dams lies in the computation of arch deformations, the author developed a method using only vertical cantilevers as principal structures, resolving the internal forces and moments from the equality in displacements of lines on radial sections of adjacent cantilevers. The internal forces and moments in radial sections have been considered as loads, estimated by trial. A convenient choice of coordinate systems with one axis parallel to the axis of the cantilevers simplified the equations.

The use of self-balancing loads and twisted structures may also be applied to the calculation of structures other than dams, as, for example, slabs of various shapes.

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